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Local symmetries in systems with constraints

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Abstract. For arbitrary systems with first-class constraints the local gauge transformations are constructed in phase and configuration spaces, i.e. a method for obtaining symmetry transformations in the second Noether's theorem is given.

1. Introduction

The approach Dirac proposed for describing systems with constraints [1] has recently attracted renewed attention in view of the fundamental role of gauge theories in elementary particle physics because these theories belong to the class of degenerate theories. The Dirac hypothesis has been under discussion for a long time and according to it, all the first-class constraints are generators of gauge transformations. In the existing literature there are many divergent opinions, some of them [2, 3] totally reject this hypothesis and some of them completely accept it [4–7], which signifies that in the general case no method yet exists for finding gauge transformations in theories with constraints. Knowledge of the explicit form of gauge transformations is necessary in many cases, for instance, in BRST [8] and $Sp(2)$ [9] quantization, for deriving improper conservation laws and for studying the connection between various gauges.

Gauge transformations were constructed by two approaches (however, in the general case the problem was not solved). One of them [6] is based on a generalized Hamiltonian H_E that is a sum of the canonical Hamiltonian and all the first-class constraints with their Lagrange multipliers; there the phase space is formally extended by assuming the Lagrange multipliers to be extra coordinates. This extension is required for removing the terms proportional to the Lagrange multipliers from the action variation by ascribing the corresponding transformations to the multipliers. This approach differs from the Dirac approach and, moreover, the space thus extended has no symplectic structure of the phase space. The other approach [3, 10–12] (without extending the phase space) also did not permit one to obtain gauge transformations in the general case. The reason is that the group structure of the generator of gauge transformations was given *a priori*: the number of arbitrary parameters was fixed beforehand (it was equal to the number of primary first-class constraints), which did not follow from the Dirac hypothesis.

In our earlier papers [4, 5], we suggested a method of constructing infinitesimal gauge transformations on the basis of the variational principle for the action. We proceeded in accordance with the Dirac hypothesis and on the algebra of constraints we imposed the restriction consisting in that the Poisson brackets of primary constraints with all constraints are linear combinations of primary constraints. Then it is natural to ask to what extent this restriction reduces the class of theories for which gauge transformations can be constructed, and what is the nature of degeneracy of Lagrangians, because there are examples [11–13] when this restriction is broken up. Note that the mentioned restriction on the constraints applies also to the aforecited articles. Moreover, these approaches do not embrace the cases when higher derivatives are present in the symmetry transformation law. The latter has also to do with the Lagrangian formalism [14].

In this paper, following the method applied in [3] (i.e. requiring the transformed coordinates to be solutions of the Hamiltonian equations of motion) and the Dirac hypothesis, we derive infinitesimal gauge transformations in phase and configuration space for arbitrary degenerated Lagrangians. We also show that the difficulty due to restriction on the algebra of constraints can be removed by passing to an equivalent set of constraints and that the degeneracy of Lagrangians stems from their being gauge-invariant. (We will consider only first-class constraints as only these constraints are responsible for gauge degrees of freedom [1].) The method will be applied to examples not yet solved [11].

2. Definitions, derivation of infinitesimal gauge transformations, algebra of constraints

Subsequent considerations may be extended to field theory, but here, for simplicity, we restrict ourselves to a system with a finite number of the degrees of freedom described by a degenerate Lagrangian $L(q, \dot{q})$, where $q = (q_1, \ldots, q_N)$ and $\dot{q} = \mathrm{d}q/\mathrm{d}t = (\dot{q}_1, \ldots, \dot{q}_N)$ are generalized coordinates and velocities, respectively. Degeneracy of the Lagrangian implies that

$$\operatorname{rank}\left\| \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j} \right\| = R < N \qquad i, j = 1, \ldots, N. \tag{1}$$

To pass into the Hamiltonian formalism, we introduce the momentum variables

$$p_i(q, \dot{q}) = \frac{\partial L}{\partial \dot{q}_i}$$

which are not all independent due to condition (1). As a result, there appear $N - R$ relationships in phase space:

$$\phi^1_\alpha(q, p) \approx 0 \qquad \alpha = 1, \ldots, N - R. \tag{2}$$

By the Dirac terminology, ϕ^1_α in (2) are primary constraints and $\approx$ means weak equality.

The Hamiltonian equations of motion are written in terms of the standard Poisson brackets as follows [1]:

$$\dot{q}_i = \{q_i, H_{\mathrm{T}}\} \qquad \dot{p}_i = \{p_i, H_{\mathrm{T}}\} \qquad \phi^1_\alpha(q, p) \approx 0 \tag{3}$$

where the total Hamiltonian H_{T} is

$$H_{\mathrm{T}} = H_{\mathrm{c}} + u_\alpha \phi^1_\alpha. \tag{4}$$

In (4) H_{c} is the canonical Hamiltonian and u_α are arbitrary functions of time.

For the system of equations (3) to be self-consistent, the primary constraints should be conserved in time. As a result, there arise secondary constraints $\phi^2_\alpha(q, p) \approx 0$ that should also be conserved in time and lead to constraints of the next stage. This process is continued up to trivial fulfillment of the conditions of stationarity which occur at a certain stage M_α. Following Dirac, we denote the whole set of constraints, both primary and secondary of all stages, as follows

$$\phi_\alpha^{m_\alpha} \qquad \alpha = 1, \ldots, N-R \qquad m_\alpha = 1, 2, \ldots, M_\alpha. \tag{5}$$

We assume that the system (5) is a complete set of independent functions [1].

In accordance with the Dirac hypothesis, we look for infinitesimal gauge transformations in the form

$$\begin{aligned} q_i' &= q_i + \delta q_i \qquad & \delta q_i &= \{q_i, G\} \\ p_i' &= p_i + \delta p_i \qquad & \delta p_i &= \{p_i, G\} \end{aligned} \tag{6}$$

where the generating function G is given by

$$G = \varepsilon_\alpha^{m_\alpha} \phi_\alpha^{m_\alpha} \qquad \alpha = 1, \ldots, N-R \qquad m_\alpha = 1, \ldots, M_\alpha \tag{7}$$

with $\varepsilon_\alpha^{m_\alpha}$ being arbitrary functions of time.

As in [3, 7, 11, 12], we require the transformed quantities q_i' and p_i' defined by (6) to be solutions of the Hamiltonian equations of motion. Let us use in the following the statement proved in [7] (p 2729). Consider a function $\psi(q, p, t)$ which is supposed to be a first-class function and satisfies the relation:

$$\{\psi, H_c\} + u_\alpha(t)\{\psi, \phi_\alpha^1\} + \frac{\partial \psi}{\partial t} = \omega_\alpha(q, p, t)\phi_\alpha^1(q, p) \tag{8}$$

where ω_α are functions of q, p and t. Then the infinitesimal canonical transformation generated by $\psi(q, p, t)$ maps the solution $[q(t), p(t), u(t)]$ into another trajectory $[q'(t), p'(t), u'(t)]$:

$$q_i'(t) = q_i(t) + \epsilon \frac{\partial \psi}{\partial p_i}(q(t), p(t), t) \tag{9}$$

$$p_i'(t) = p_i(t) - \epsilon \frac{\partial \psi}{\partial q_i}(q(t), p(t), t) \tag{10}$$

$$u_\alpha'(t) = u_\alpha(t) + \epsilon \omega_\alpha(q(t), p(t), t) \tag{11}$$

which is also an extremal of action S and satisfies the equations of motion (3) with the Hamiltonian H_T':

$$\begin{aligned} H_T'(q, p, t) &= H_T(q, p, t) + \epsilon \left[\frac{\partial \psi}{\partial t}(q, p, t) + \{\psi, H_T\} \right] \\ &= H_c(q, p) + u_\alpha'(t)\phi_\alpha^1(q, p). \end{aligned} \tag{12}$$

The condition of ψ being a first-class function guarantees that ψ generates transformations of the points (q, p) into points (q', p') of the same manifold. In conformity with the Dirac hypothesis, we search for ψ in the form (7), i.e. $\psi \equiv G$.

We recall that we consider only first-class constraints, which implies the following relations

$$\{\phi_\alpha^{m_\alpha}, \phi_\beta^{m_\beta}\} = f_{\alpha}^{\ m_\alpha}{}_{\beta}^{\ m_\beta}{}_{\gamma}^{\ m_\gamma}\, \phi_\gamma^{m_\gamma} \tag{13}$$

$$\{\phi_\sigma^{m_\sigma}, H_c\} = g_{\sigma}^{\ m_\sigma}{}_{\tau}^{\ m_\tau}\, \phi_\tau^{m_\tau} \tag{14}$$

$$m_{\alpha,\beta,\gamma,\sigma} = 1, \ldots, M_{\alpha,\beta,\gamma,\sigma} \qquad m_\tau = 1, \ldots, m_\sigma + 1.$$

(Here and in what follows, summation runs over repeated upper and lower indices.) Using these relations and the function G defined by (7) we rewrite equation (8) in the form

$$[(\dot\varepsilon_\alpha^{m_\alpha} + g_{\beta}^{\ m_\beta}{}_{\alpha}^{\ m_\alpha}\, \varepsilon_\beta^{m_\beta})\phi_\alpha^{m_\alpha} + (\dot\varepsilon_\alpha^{1} + g_{\beta}^{\ m_\beta}{}_{\alpha}^{\ 1}\, \varepsilon_\beta^{m_\beta} - u_\gamma f_{\gamma}^{\ 1}{}_{\beta}^{\ m_\beta}{}_{\alpha}^{\ 1}\, \varepsilon_\beta^{m_\beta})\phi_\alpha^1$$
$$+ u_\gamma f_{\beta}^{\ m_\beta}{}_{\gamma}^{\ 1}{}_{\alpha}^{\ m_\alpha}\, \varepsilon_\beta^{m_\beta}\phi_\alpha^{m_\alpha}]_{\phi_\alpha^1 \approx 0} = 0 \qquad m_\alpha \geqslant 2 \qquad m_\beta \geqslant m_\alpha - 1. \tag{15}$$

Owing to the constraints being independent, the equality (15) can be satisfied if the coefficients of secondary constraints of all stages vanish, i.e.

$$(\dot\varepsilon_\alpha^{m_\alpha} + g_{\beta}^{\ m_\beta}{}_{\alpha}^{\ m_\alpha}\, \varepsilon_\beta^{m_\beta}) + u_\gamma f_{\beta}^{\ m_\beta}{}_{\gamma}^{\ 1}{}_{\alpha}^{\ m_\alpha}\, \varepsilon_\beta^{m_\beta} = 0 \qquad m_\alpha \geqslant 2. \tag{16}$$

This equality cannot be satisfied by any selection of functions $\varepsilon_\alpha^{m_\alpha}$ because the Lagrange multipliers $u_\gamma(t)$ are arbitrary. However, when

$$f_{\beta}^{\ m_\beta}{}_{\gamma}^{\ 1}{}_{\alpha}^{\ m_\alpha} = 0 \qquad \text{for } m_\alpha \geqslant 2 \tag{17}$$

we obtain [4]

$$\dot\varepsilon_\alpha^{m_\alpha} + g_{\beta}^{\ m_\beta}{}_{\alpha}^{\ m_\alpha}\, \varepsilon_\beta^{m_\beta} = 0 \qquad m_\beta \geqslant m_\alpha - 1. \tag{18}$$

Note that the condition (17) is equivalent to the relation

$$\{\phi_\beta^1, \phi_\gamma^{m_\gamma}\} = f_{\beta}^{\ 1}{}_{\gamma}^{\ m_\gamma}{}_{\alpha}^{\ 1}\, \phi_\alpha^1. \tag{19}$$

In our previous papers [4, 5], on the basis of variational principle, we have derived the relation (18) between the parameters $\varepsilon_\alpha^{m_\alpha}$ for systems with constraints obeying the condition (19).

The system of equations (18) is not complete; the number of unknown functions exceeds the number of equations by the number of primary constraints. Therefore, introducing arbitrary functions ε_α in the amount equal to that of primary constraints and applying the iteration procedure to (18), we can express all $\varepsilon_\alpha^{m_\alpha}$ in terms of the introduced functions and $g_{\beta}^{\ m_\beta}{}_{\alpha}^{\ m_\alpha}$ and their derivatives [5]:

$$\varepsilon_\alpha^{m_\alpha} = B_{\alpha}^{\ m_\alpha}{}_{\beta}^{\ m_\beta}\, \varepsilon_\beta^{(M_\alpha - m_\beta)} \qquad m_\beta = m_\alpha, \ldots, M_\alpha \tag{20}$$

(here summation runs also over m_β), where

$$\varepsilon_\beta^{(M_\alpha - m_\beta)} \equiv \frac{\mathrm{d}^{M_\alpha - m_\beta}}{\mathrm{d}t^{M_\alpha - m_\beta}}\varepsilon_\beta(t) \qquad \varepsilon_\beta(t) \equiv \varepsilon_\beta^{M_\beta}$$

and $B^{m_\alpha\ m_\beta}_{\alpha\ \beta}$ are, generally speaking, functions of q and p and their derivatives up to the order $M_\alpha - m_\alpha - 1$. Then the generating function of gauge transformations G assume the form

$$G = B^{m_\alpha\ m_\beta}_{\alpha\ \beta}\phi^{m_\alpha}_\alpha \varepsilon^{(M_\alpha - m_\beta)}_\beta \qquad m_\beta = m_\alpha, \dots, M_\alpha. \tag{21}$$

Owing to the derivatives of q and p with respect to time being present in $B^{m_\alpha\ m_\beta}_{\alpha\ \beta}$, the Poisson brackets are not determined in the transformations (6). However, this problem was solved in our previous paper [5] where it was shown that gauge transformations generated by the generating function G (21) are canonical in extended phase space. The action is invariant under these transformations and the corresponding gauge transformations in the configuration space (see Appendix A)[1] The number of arbitrary functions the function G is dependent on equals the number of primary constraints. As can be seen from formula (21), the transformation law may include both arbitrary functions $\varepsilon_\alpha(t)$ and their derivatives up to and including order $M_\alpha - 1$; the highest derivatives $\varepsilon^{(M_\alpha - 1)}_\alpha$ should always be present.

Note that derivation of formula (21) presents no extra difficulties and does not require further assumptions, as compared with the Dirac approach.

Also we mention that since the first-class constraints compose a quasi-algebra [15], the condition (19) means that primary constraints represent the ideal of that quasi-algebra. Below, we show that for arbitrary Lagrangians (even when condition (19) does not hold) we can always pass to an equivalent set of constraints for which the condition (19) will be valid.

3. Gauge transformations for an arbitrary degenerate Lagrangian

To generalize the method of construction of gauge transformations to an arbitrary degenerate Lagrangian, it is necessary to analyse the situation when the condition (19) is not fulfilled. To this end, let us recall the inherent arbitrariness of the Dirac Hamiltonian formalism. When there is a complete set of constraints defined by the Dirac procedure ($\phi^{m_\alpha}_\alpha$, $\alpha = 1, \dots, N-R$; $m_\alpha = 1, \dots, M_\alpha$) and functionally independent, we can always pass to an equivalent set of constraints by the transformation

$$\bar{\phi}^{m_\beta}_\beta = C^{m_\beta\ m_\alpha}_{\beta\ \alpha}\phi^{m_\alpha}_\alpha \tag{22}$$

where

$$\det \| C^{m_\beta\ m_\alpha}_{\beta\ \alpha} \|_\Sigma \neq 0 \tag{23}$$

i.e. this determinant is not zero on the surface Σ given by the complete set of constraints.

Now consider a particular case of the transformation (22) when primary constraints remain unchanged, i.e.

$$C^{1\ m_\alpha}_{\beta\ \alpha} = \delta_{\beta\alpha} \qquad \text{for any } m_\alpha.$$

[1] The corresponding gauge transformations in the configuration space are defined as follows:

$$\delta q_i(t) = \{q_i(t), G\}|_{p=\frac{\partial L}{\partial \dot q}} \qquad \delta \dot q(t) = \frac{\mathrm{d}}{\mathrm{d}t}\delta q(t).$$

It is not difficult to see that taking account of (13) and (14) we obtain

$$\{\phi_\alpha^1, \bar{\phi}_\beta^{m_\beta}\} = [\{\phi_\alpha^1, C_{\beta\gamma}^{m_\beta m_\gamma}\} + f_{\alpha\delta\gamma}^{1\,m_\delta\,m_\gamma} C_{\beta\delta}^{m_\beta m_\delta}]\phi_\gamma^{m_\gamma} + f_{\alpha\delta\gamma}^{1\,m_\delta\,1} C_{\beta\delta}^{m_\beta m_\delta}\phi_\beta^1$$
$$m_\beta, m_\delta, m_\gamma \geqslant 2. \tag{24}$$

From the expression (24) it is clear that if we could choose $C_{\beta\gamma}^{m_\beta m_\gamma}$ so that the coefficients of secondary constraints vanish

$$\{\phi_\alpha^1, C_{\beta\gamma}^{m_\beta m_\gamma}\} + f_{\alpha\delta\gamma}^{1\,m_\delta\,m_\gamma} C_{\beta\delta}^{m_\beta m_\delta} = 0 \tag{25}$$

condition (19) will be valid for the new set of constraints $\bar{\phi}_\beta^{m_\beta}$. Thus, for $C_{\beta\gamma}^{m_\beta m_\gamma}$ we have derived a system of linear inhomogeneous equations in the first-order partial derivatives (25). This system can be shown to be fully integrable. The condition of integrability for systems of the type (25) is as follows [16]:

$$\{\phi_\sigma^1, \{\phi_\alpha^1, C_{\beta\gamma}^{m_\beta m_\gamma}\}\} - \{\phi_\alpha^1, \{\phi_\sigma^1, C_{\beta\gamma}^{m_\beta m_\gamma}\}\} = 0. \tag{26}$$

Using (25), properties of the Poisson brackets and making some transformations we rewrite relation (26) in the form

$$[\{\phi_\alpha^1, f_{\sigma\delta\gamma}^{1\,m_\delta\,m_\gamma}\} - f_{\alpha\delta\tau}^{1\,m_\delta\,m_\tau} f_{\sigma\tau\gamma}^{1\,m_\tau\,m_\gamma} - \{\phi_\sigma^1, f_{\alpha\delta\gamma}^{1\,m_\delta\,m_\gamma}\} + f_{\sigma\delta\tau}^{1\,m_\delta\,m_\tau} f_{\alpha\tau\gamma}^{1\,m_\tau\,m_\gamma}] C_{\beta\delta}^{m_\beta m_\delta} = 0$$
$$m_\beta, m_\delta, m_\tau \geqslant 2. \tag{27}$$

Utilizing the Jacobi identity

$$\{\phi_\alpha^1, \{\phi_\sigma^1, \phi_\beta^{m_\beta}\}\} + \{\phi_\beta^{m_\beta}, \{\phi_\alpha^1, \phi_\sigma^1\}\} + \{\phi_\sigma^1, \{\phi_\beta^{m_\beta}, \phi_\alpha^1\}\} = 0 \qquad m_\beta \geqslant 2$$

and relation (13) we obtain

$$[\{\phi_\alpha^1, f_{\sigma\delta\gamma}^{1\,m_\delta\,m_\gamma}\} - f_{\alpha\delta\tau}^{1\,m_\delta\,m_\tau} f_{\sigma\tau\gamma}^{1\,m_\tau\,m_\gamma} - \{\phi_\sigma^1, f_{\alpha\delta\gamma}^{1\,m_\delta\,m_\gamma}\} + f_{\sigma\delta\tau}^{1\,m_\delta\,m_\tau} f_{\alpha\tau\gamma}^{1\,m_\tau\,m_\gamma}]\phi_\gamma^{m_\gamma}$$
$$= \{\{\phi_\alpha^1, \phi_\sigma^1\}, \phi_\delta^{m_\delta}\} \qquad m_\beta \geqslant 2 \qquad m_\gamma, m_\delta, m_\tau \geqslant 1. \tag{28}$$

Note that the Poisson brackets between primary constraints may, without loss of generality, be considered to be strictly zero in the whole phase space. As every primary constraint contains at least one momentum variable, there always exist canonical transformations transforming the primary constraints into new momentum variables (see below). Therefore, the expressions in square brackets in front of the constraints $\phi_\gamma^{m_\gamma}$ on the left-hand side of the identity (28), being coefficients of the functionally independent quantities, each disappear separately. As condition (27) contains the same coefficients of $C_{\beta\delta}^{m_\beta m_\delta}$, it is satisfied identically, which proves the system of equations (25) to be fully integrable. Therefore, there always exists a set of constraints equivalent to the initial set for which condition (19) holds valid.

Now we shall describe the way of passing to, at least, one separated set of equivalent constraints $\bar{\phi}_\alpha^{m_\alpha}$ when all the primary constraints are momentum variables. This can be done by the iteration procedure provided that we take into account the property of primary constraints

$$\{\phi_\alpha^1, \phi_\beta^1\} = f_{\alpha\ \beta\ \gamma}^{1\ 1\ 1}\phi_\gamma^1$$

that follows from the stationarity condition for ϕ_α^1 and from the fact that we are dealing only with first-class constraints. There always exist canonical transformations of the form [17, 18]

$$\bar{P}_1 = \phi_1^1(q, p) \qquad \{\bar{Q}_1, \bar{P}_1\} = 1 \qquad \{\bar{Q}_\sigma, \bar{P}_\tau\} = \delta_{\sigma\tau}$$
$$\{\bar{P}_1, \bar{P}_\tau\} = \{\bar{Q}_1, \bar{P}_\tau\} = \{\bar{P}_1, \bar{Q}_\tau\} = \{\bar{Q}_1, \bar{Q}_\tau\} = 0 \qquad \sigma, \tau = 2, \ldots, N. \tag{29}$$

(The bar over a letter means the first stage of the iteration procedure.) All the remaining primary constraints assume the form

$$\Phi_\alpha^1(\bar{Q}, \bar{P}) = \phi_\alpha^1(q(\bar{Q}, \bar{P}), p(\bar{Q}, \bar{P}))|_{\bar{P}_1=0} \qquad \alpha = 2, \ldots, N-R.$$

In view of the transformation being canonical, we can write

$$\{\bar{P}_1, \Phi_\alpha^1\} = -\frac{\partial \Phi_\alpha^1}{\partial \bar{Q}_1} = \bar{f}_{1\ \alpha\ \gamma}^{1\ 1\ 1}\Phi_\gamma^1 \qquad \alpha, \gamma \geqslant 2$$

with Φ_α^1 having the structure [18]

$$\Phi_\alpha^1 = \bar{D}_{\alpha\ \gamma}^{1\ 1}\bar{\Phi}_\gamma^1 \qquad \det \bar{D}|_\Sigma \neq 0 \tag{30}$$

and obeying the conditions

$$\frac{\partial \bar{\Phi}_\gamma^1}{\partial \bar{Q}_1} = \frac{\partial \bar{\Phi}_\gamma^1}{\partial \bar{P}_1} = 0 \qquad \gamma \geqslant 2.$$

As all the constraints $\bar{\Phi}_\gamma^1$ do not depend upon $\bar{Q}_1$ and $\bar{P}_1$, we perform an analogous procedure for the constraint $\bar{\Phi}_2^1$ in the $2N-2$-dimensional subspace $(\bar{Q}_\sigma, \bar{P}_\sigma)(\sigma = 2, \ldots, N)$, i.e. without affecting $\bar{Q}_1$ and $\bar{P}_1$. Then the constraints $\bar{\bar{\Phi}}_\alpha^1(\alpha = 3, \ldots, N-R)$ arising in a formula analogous to formula (30) are independent of $\bar{Q}_1$, $\bar{P}_1$ and $\bar{\bar{Q}}_2$, $\bar{\bar{P}}_2$. Next, making this procedure step-by-step $N-R-2$ times we finally obtain the primary constraints to be momenta, and therefore they commute with each other (final momenta and coordinates will be denoted by Q_α and P_α, respectively, $\alpha = 1, \ldots, N-R$). All secondary constraints will then assume the form

$$\Phi_\alpha^{m_\alpha}(Q, P) = \phi_\alpha^{m_\alpha}(q(Q, P), p(Q, P))|_{P_\alpha=0} \qquad \alpha = 1, \ldots, N-R \qquad m_\alpha = 2, \ldots, M_\alpha.$$

As the transformations are canonical, we can write

$$\{P_\alpha, \Phi_\beta^{m_\beta}\} = -\frac{\partial \Phi_\beta^{m_\beta}}{\partial \bar{Q}_\alpha} = f_{\alpha\ \beta\ \gamma}^{1\ m_\beta\ m_\gamma}\Phi_\gamma^{m_\gamma}$$

with $\Phi_\alpha^{m_\alpha}$ having the structure [18]

$$\Phi_\alpha^{m_\alpha} = A_{\alpha\ \beta}^{m_\alpha\ m_\beta}\tilde{\Phi}_\beta^{m_\beta} \qquad \det A|_\Sigma \neq 0 \tag{31}$$

and obeying the conditions

$$\frac{\partial \tilde{\Phi}_\alpha^{m_\alpha}}{\partial Q_\beta} = \frac{\partial \tilde{\Phi}_\alpha^{m_\alpha}}{\partial P_\beta} = 0 \qquad \alpha, \beta = 1, \ldots, N - R \qquad m_\alpha \geqslant 2.$$

The set of constraints thus constructed (primary constraints being momenta and secondary $\tilde{\Phi}_\alpha^{m_\alpha}$) satisfies condition (19) with vanishing right-hand side, i.e. we have derived the searched for set of constraints. Note that $(A^{-1})_{\alpha\ \beta}^{m_\alpha m_\beta}$ in (31) is a solution to the system of equations (25).

So, we may conclude that the difficulty associated with the condition (19) not being valid for a certain degenerate Lagrangian can be overcome by passing to an equivalent set of constraints. Therefore, the method we proposed earlier for constructing gauge transformations [4, 5] is applicable in the general case.

4. Examples

1. Consider the Lagrangian [11]

$$L = \frac{1}{\alpha}\dot{y} \cdot (\dot{x} + \beta y) \tag{32}$$

where α and β are scalar coordinates; x and y are n-dimensional vectors.

In the Hamiltonian formalism there are the following primary constraints

$$\phi_1^1 = p_\alpha \approx 0 \qquad \phi_2^1 = p_\beta \approx 0$$

and the total Hamiltonian

$$H_{\mathrm{T}} = \alpha p_x \cdot p_y - \beta y \cdot p_x + u_1 p_\alpha + u_2 p_\beta.$$

From the condition of self-consistency of the theory we obtain two secondary and one tertiary constraint:

$$\phi_1^2 = -p_x \cdot p_y \approx 0 \qquad \phi_2^2 = y \cdot p_x \approx 0 \qquad \phi_1^3 = -\beta p_x^2 \approx 0.$$

It may be verified directly that all these constraints are of the first class and they do not obey condition (19)

$$\{\phi_2^1, \phi_1^3\} = -\frac{1}{\beta}\phi_1^3.$$

As the primary constraints in the example are momentum variable, we can turn them into an equivalent set of constraints by formula (31):

$$\bar{\phi}_1^1 = \phi_1^1 = p_\alpha \qquad \bar{\phi}_2^1 = \phi_2^1 = p_\beta$$

$$\bar{\phi}_1^2 = \phi_1^2 = -p_x \cdot p_y \qquad \bar{\phi}_2^2 = \phi_2^2 = y \cdot p_x \qquad \bar{\phi}_1^3 = -\frac{1}{\beta}\phi_1^3 = p_x^2$$

which do satisfy condition (19): $\{\bar{\phi}_2^1, \bar{\phi}_1^3\} = 0$.

All $\bar{g}^{m_\beta}_{\beta}{}^{m_\alpha}_{\alpha}$ in formula (14) for the Poisson brackets of these constraints with the canonical Hamiltonian vanish except for

$$\bar{g}^{1}_{1}{}^{2}_{1} = \bar{g}^{1}_{2}{}^{2}_{2} = 1 \qquad \bar{g}^{2}_{1}{}^{3}_{1} = -\beta \qquad \bar{g}^{2}_{2}{}^{3}_{1} = \alpha.$$

Then the system of equations (18) takes the form

$$\begin{aligned} &\dot{\varepsilon}^3_1 - \beta\varepsilon^2_1 + \alpha\varepsilon^2_2 = 0 \\ &\dot{\varepsilon}^2_2 + \varepsilon^1_2 = 0 \\ &\dot{\varepsilon}^2_1 + \varepsilon^1_1 = 0. \end{aligned} \tag{33}$$

With the redefinition $\varepsilon^3_1 \equiv \varepsilon_1$ and $\varepsilon^2_2 \equiv \varepsilon_2$ we obtain

$$\varepsilon^2_1 = \frac{1}{\beta}(\dot{\varepsilon}_1 + \alpha\varepsilon_2) \qquad \varepsilon^1_2 = -\dot{\varepsilon}_2 \qquad \varepsilon^1_1 = -\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{\varepsilon}_1 + \alpha\varepsilon_2}{\beta}\right).$$

As the parameters in the generating function depend on $\dot{\alpha}$ and $\dot{\beta}$, we can derive the 'well-defined' Poisson brackets by applying the procedure of extending the phase space described in our previous paper [5]. It is sufficient to make the following extension of the phase space: introduce the coordinates

$$\begin{aligned} &q_{1\ i} = \begin{cases} \alpha & i = 1 \\ \beta & i = 2 \\ x_i & i = 3, \ldots, n+2 \\ y_i & i = n+3, \ldots, 2n+2 \end{cases} \\ &q_{2\ i} \equiv \dot{q}_{1\ i} \end{aligned} \tag{34}$$

whereas momenta follow from the definition of momentum variables in the theories with higher derivatives [18–20] (see formula (49) in the appendix: in our case, $K = 2$, $r = 1, 2$, $i = 1, \ldots, 2n+2$). Note that by this definition, apart from the existing primary constraints $\phi^1_1 = p_{1\ 1} = p_\alpha$ and $\phi^1_2 = p_{1\ 2} = p_\beta$, there appear extra momenta $\phi^1_i = p_{2\ i} = 0$. Then, in extended phase space we have

$$\bar{H}_{\mathrm{c}} = H_{\mathrm{c}}(q_{1\ i}, p_{1\ i}) \qquad \bar{H}_{\mathrm{T}} = H_{\mathrm{T}} + v_i p_{2\ i}$$

where v_i are the Lagrange multipliers and H_{c} does not contain $q_{2\ i}$ and $p_{2\ i}$. As the extra coordinates and momenta which extend the phase space enter into $\bar{H}_{\mathrm{T}}$ as separate terms, the structure of algebra of constraints and the system of equations (33) are not changed. Due to the definition (34) we can rewrite the generating function G (7) in the form

$$\begin{aligned} G = &\left[-\ddot{\varepsilon}_1 + \frac{q_{2\ 2}}{q_{1\ 2}}\dot{\varepsilon}_1 - q_{1\ 1}\dot{\varepsilon}_2 + \left(\frac{q_{2\ 2}}{q_{1\ 2}} - q_{2\ 1}\right)\varepsilon_2\right]\frac{p_{1\ 1}}{q_{1\ 2}} - \dot{\varepsilon}_2 p_{1\ 2} \\ &+ \sum_{m=3}^{n+2}\left[\frac{1}{q_{1\ 2}}(\dot{\varepsilon}_1 + q_{1\ 1}\varepsilon_2)p_{1\ m}p_{1\ m+n} - \varepsilon_1(p_{1\ m})^2 - \varepsilon_2 q_{1\ m+n}p_{1\ m}\right] \end{aligned} \tag{35}$$

from which it is clear that the corresponding transformations are canonical in the expanded phase space.

Using the explicit form of coordinate transformations in the configuration space (see footnote 1), we can write

$$\delta\alpha = -\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot\varepsilon_1 + \alpha\varepsilon_2}{\beta}\right) \qquad \delta\beta = -\dot\varepsilon_2 \qquad \delta y = -\frac{\dot y}{\alpha\beta}(\dot\varepsilon_1 + \alpha\varepsilon_2)$$

$$\delta x = -\frac{1}{\alpha\beta}(\dot x + \beta y)(\dot\varepsilon_1 + \alpha\varepsilon_2) + 2\frac{\dot y}{\alpha}\varepsilon_1 + y\varepsilon_2$$

and it is not difficult to obtain that

$$\delta L = \frac{\mathrm{d}}{\mathrm{d}t}\left[-\frac{\dot y}{\alpha^2\beta}(\dot x + \beta y)(\dot\varepsilon_1 + \alpha\varepsilon_2) + \left(\frac{\dot y}{\alpha}\right)^2\varepsilon_1\right]$$

i.e. the action is invariant under the gauge transformations we have derived.

The transformation law (35) in this example is consistent with the requirements we have discussed above, i.e. it is a two-parameter transformation as there are only two primary constraints, and it contains $\ddot\varepsilon_1$ and $\dot\varepsilon_2$, i.e. $M_1 = 3$ and $M_2 = 2$. Note that the Noether transformations derived for the Lagrangian (32) in [11] are a particular case of our transformations (they are one-parameter transformations).

2. *Polyakov's string* [11, 12 and 21]. The Lagrangian density is given by

$$\begin{aligned} L &= \frac{\sqrt{-h}}{2}h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu \\ &= \frac{-1}{2\sqrt{-h}}(h_{11}\dot X^2 - 2h_{01}(\dot X X') + h_{00}X'^2) \end{aligned} \tag{36}$$

where $\alpha, \beta = 0, 1$, $\mu = 0, 1, \ldots, D-1$ and $\partial_1 X \equiv \dot X$, $\partial_0 X \equiv X'$.

The canonical momenta are

$$P_\mu = \frac{\partial L}{\partial \dot X^\mu} = \frac{-1}{\sqrt{-h}}(h_{11}\dot X_\mu - h_{01}X'_\mu) \qquad \Pi^{\alpha\beta} = \frac{\partial L}{\partial \dot h_{\alpha\beta}} = 0 \tag{37}$$

so the canonical Hamiltonian is

$$H_c = -\frac{\sqrt{-h}}{h_{11}}H + \frac{h_{01}}{h_{11}}T \tag{38}$$

where $H = \frac{1}{2}(P^2 + X'^2)$ and $T = (PX')$. There are three primary constraints:

$$\phi_1 = \Pi_{00} \qquad \phi_2 = \Pi_{01} \qquad \phi_3 = \Pi_{11}. \tag{39}$$

Conservation in time of the primary constraints gives

$$\begin{aligned} \dot\Pi_{00} &= \frac{-1}{2\sqrt{-h}}H \\ \dot\Pi_{01} &= \frac{h_{01}}{\sqrt{-h}h_{11}}H - \frac{1}{h_{11}}T \\ \dot\Pi_{11} &= \frac{-1}{\sqrt{-h}}\left(\frac{h_{00}}{2h_{11}} - \frac{h}{h_{11}^2}\right)H + \frac{h_{01}}{h_{11}^2}T. \end{aligned} \tag{40}$$

Following the above Dirac scheme of generating the constraints, one should declare, for example, the right-hand sides of first two equalities (40) to be secondary constraints:

$$\phi_1^2 = \frac{-1}{2\sqrt{-h}} H \qquad \phi_2^2 = \frac{h_{01}}{\sqrt{-h} h_{11}} H - \frac{1}{h_{11}} T. \tag{41}$$

It is seen that the right-hand side of the third equality (40) is expressed linearly through the constraints (41) and, therefore, does not give a new secondary constraint. However, as one can verify, the constraints (39) and (41) do not obey condition (19). Therefore, resorting to the procedure described in section 3, we introduce a new set of the secondary constraints:

$$\bar{\phi}_\sigma^2 = C_{\sigma\tau} \phi_\tau^2 \qquad \sigma, \tau = 1, 2 \tag{42}$$

where the coefficients $C_{\sigma\tau}$ satisfy the equation system (25), the concrete form of which is not adduced here because of the formulae being cumbersome, but we immediately give its solution:

$$C_{11} = -2\sqrt{-h} \qquad C_{12} = 0 \qquad C_{21} = -2h_{01} \qquad C_{22} = -h_{11}. \tag{43}$$

Then we obtain

$$\bar{\phi}_1^2 = H \qquad \bar{\phi}_2^2 = T. \tag{44}$$

The conditions of conservation in time of the constraints (44) do not give the tertiary constraints.

Note that exactly such secondary constraints (44) right from the start (maybe, for reasons of simplicity) are selected in [11] and [12] and therefore satisfy condition (19), hence the transformations of coordinates in phase space which are obtained in the above work are the symmetry transformations (unlike the consideration of the previous example in the same work [11]).

To derive the transformations of coordinates by our method, we move to another equivalent set of the primary constraints, as in [11] and [12], although after the choice of the secondary constraints in the form to satisfy condition (19) one might at once apply our method (without passing to an equivalent set of the primary constraints). So, define

$$\begin{aligned}
\bar{\phi}_1^1 &= \frac{2\sqrt{-h} h_{01}}{h_{00}} \Pi_{01} + \frac{2\sqrt{-h} h_{11}}{h_{00}} \Pi_{11} \\
\bar{\phi}_2^1 &= \frac{2h_{01}^2 - h_{00} h_{11}}{h_{00}} \Pi_{01} + \frac{2h_{01} h_{11}}{h_{00}} \Pi_{11} \\
\bar{\phi}_3^1 &= h_{00}\Pi_{00} + h_{01}\Pi_{01} + h_{11}\Pi_{11}.
\end{aligned} \tag{45}$$

Using the relation (14) for the new set of the constraints (44) and (45), we obtain the functions $g\,{}^{m_\sigma}_{\sigma}\,{}^{m_\tau}_{\tau}\,\phi_\tau^{m_\tau}$:

$$\begin{aligned}
g_1^1{}_1^2 &= g_2^1{}_2^2 = \delta(\sigma' - \sigma) \\
g_1^2{}_1^2 &= g_2^2{}_2^2 = 2\partial_\sigma \left(\frac{h_{01}}{h_{11}} \right) \delta(\sigma' - \sigma) + \frac{h_{01}(\sigma)}{h_{11}(\sigma)} \delta'_{\sigma'}(\sigma' - \sigma) \\
g_1^2{}_2^2 &= g_2^2{}_1^2 = -2\partial_\sigma \left(\frac{\sqrt{-h}}{h_{11}} \right) \delta(\sigma' - \sigma) + \frac{\sqrt{-h(\sigma)}}{h_{11}(\sigma)} \delta'_{\sigma'}(\sigma' - \sigma)
\end{aligned} \tag{46}$$

all the remaining $g^{m_\sigma}_{\sigma}{}^{m_\tau}_{\tau}\phi^{m_\tau}_{\tau}$ are equal to zero.

Then the system of equations (18) takes the form:

$$\begin{aligned} &\dot{\varepsilon}_1^2 - \partial_\sigma\left(\frac{h_{01}}{h_{11}}\right)\varepsilon_1^2 + \frac{h_{01}}{h_{11}}\partial_\sigma\varepsilon_1^2 - \partial_\sigma\left(\frac{\sqrt{-h}}{h_{11}}\right)\varepsilon_2^2 + \frac{\sqrt{-h}}{h_{11}}\partial_\sigma\varepsilon_2^2 + \varepsilon_1^1 = 0 \\ &\dot{\varepsilon}_2^2 + \partial_\sigma\left(\frac{\sqrt{-h}}{h_{11}}\right)\varepsilon_1^2 - \frac{\sqrt{-h}}{h_{11}}\partial_\sigma\varepsilon_1^2 - \partial_\sigma\left(\frac{h_{01}}{h_{11}}\right)\varepsilon_2^2 + \frac{h_{01}}{h_{11}}\partial_\sigma\varepsilon_2^2 + \varepsilon_2^1 = 0. \end{aligned} \tag{47}$$

With the redefinition $\varepsilon_1^2 \equiv \varepsilon_1$, $\varepsilon_2^2 \equiv \varepsilon_2$ and $\varepsilon_3^1 \equiv \varepsilon_3$ we obtain

$$\begin{aligned} \varepsilon_1^1 &= -\dot{\varepsilon}_1 + \partial_\sigma\left(\frac{h_{01}}{h_{11}}\right)\varepsilon_1 - \frac{h_{01}}{h_{11}}\partial_\sigma\varepsilon_1 + \partial_\sigma\left(\frac{\sqrt{-h}}{h_{11}}\right)\varepsilon_2 - \frac{\sqrt{-h}}{h_{11}}\partial_\sigma\varepsilon_2 \\ \varepsilon_2^1 &= -\dot{\varepsilon}_2 - \partial_\sigma\left(\frac{\sqrt{-h}}{h_{11}}\right)\varepsilon_1 + \frac{\sqrt{-h}}{h_{11}}\partial_\sigma\varepsilon_1 + \partial_\sigma\left(\frac{h_{01}}{h_{11}}\right)\varepsilon_2 - \frac{h_{01}}{h_{11}}\partial_\sigma\varepsilon_2. \end{aligned} \tag{48}$$

Inserting (44), (45) and (48) into (7), on the basis of formulae (6) we obtain the transformations of coordinates in phase space which exactly coincide with the expressions (5.12) in [12]. Then the authors of [12] go from the transformations of coordinates in the form (5.12) to those in covariant form via the introduction of a new set of the parameters. However, upon this replacement of parameters the transformations of coordinates become non-canonical because of the appearance of a term with $\dot{h}_{00}$ (the Poisson brackets of $\dot{h}_{00}$ with the variables of the phase space are not determined). Therefore, in the following one must proceed according to our method [5] and extend phase space as in the previous example. This removes all the above difficulties.

5. Conclusions

We have suggested a method of constructing gauge transformations for arbitrary degenerate Lagrangians (without restrictions on the algebra of constraints) in the generalized Hamiltonian formalism: they can be obtained explicitly on the basis of a specific form of the Lagrangian. The generating function given by (21) is derived from the requirement for the transformed quantities q_i' and p_i' (6) to be solutions of the same system of equations (3) as the initial quantities q_i and p_i. As to gauge transformations, this requirement is equivalent to the invariance of the action under these transformations or to the stationarity of the generating function (7) on the surface of the primary constraints (8), i.e. the generating function G, derived on the basis of one of them, satisfies the other two.

In our previous papers [4, 5] we constructed gauge transformations for Lagrangians with the only restriction on the algebra being constraints (19), which is satisfied by a wide class of theories. In this paper we have proved that there always exist equivalent sets of constraints for which the condition (19) holds valid. We have shown the way of transition to one of these sets when all the primary constraints are momentum variables.

The generating function (21) corresponds, in form, to the Dirac hypothesis in the sense that all the first-class constraints generate gauge transformations. The number of arbitrary functions (important parameters) which the function G depends on is equal to the number of primary constraints. Note an essential peculiarity: the transformation law contains essential parameters and their derivatives, but the leading derivative is always present and is of order one smaller than the number of stages in deriving secondary constraints by the Dirac procedure.

By the formulae in footnote 1, we have obtained the Noether transformations (i.e. with respect to which the action is invariant) in configuration space. The mechanism of the appearance of higher-order derivatives with respect to coordinates established earlier [5] in the class of theories with restriction on the algebra of constraints is now applicable in the general case.

As is known, gauge-invariant theories belong to the class of degenerate theories. In this paper we have shown that the degeneracy of theories with first-class constraints is due to their invariance under gauge transformations we have here constructed.

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Appendix

We will now show invariance of the action under the gauge transformations (6) and (21) in the phase space expanded by the method of [5]. The coordinates are defined as follows:

$$q_{1\ i} = q_i \qquad q_{s\ i} = \frac{\mathrm{d}^{s-1}}{\mathrm{d}t^{s-1}} q_i \qquad s = 2, \ldots, K \qquad i = 1, \ldots, N$$

(K equals the highest order of derivatives of q and p) and the conjugate momenta defined by the formula [17, 19]

$$p_{r\ i} = \sum_{l=r}^{K} (-1)^{l-r} \frac{\mathrm{d}^{l-r}}{\mathrm{d}t^{l-r}} \frac{\partial L}{\partial q_{r+1\ i}} \tag{49}$$

are

$$p_{1\ i} = p_i \qquad p_{s\ i} = 0 \qquad \text{for } s = 2, \ldots, K.$$

The generalized momenta for $s \geqslant 2$ are extra primary constraints.

The total Hamiltonian is of the form

$$\bar{H}_{\mathrm{T}} = H_{\mathrm{T}}(q_{1\ i}, p_{1\ i}) + \lambda_{s\ i} p_{s\ i} \qquad s \geqslant 2 \tag{50}$$

where H_{T} is given by (4) and $\lambda_{s\ i}$ are arbitrary functions of time. From (50) we may conclude that there do not appear additional secondary constraints corresponding to $p_{s\ i}$ for $s \geqslant 2$.

The set of constraints (5) in the initial phase space remains the same in extended phase space, obeys the same algebra (13), (14), and does not depend on the new coordinates and momenta. The action is of the form

$$S = \int_{t_1}^{t_2} \mathrm{d}t \, [p_{r\ i} q_{r+1\ i} + p_{K\ i} \dot{q}_{K\ i} - \bar{H}_{\mathrm{T}}] \qquad r = 1, \ldots, K-1 \tag{51}$$

and the generating function in the extended phase space is [5]

$$G = B_{\alpha\ \beta}^{m_\alpha\ m_\beta}(q_{1\ i}, q_{2\ i}, \ldots, q_{K\ i}; p_{1\ i})\phi_\alpha^{m_\alpha}(q_{1\ i}, p_{1\ i})\varepsilon_\beta^{(M_\alpha - m_\beta)}(t).$$

The coordinates and momenta are then transformed in the following way

$$\delta q_{1\ i} = \frac{\partial(B_{\alpha\ \beta}^{m_\alpha\ m_\beta}\phi_\alpha^{m_\alpha})}{\partial p_{1\ i}}\varepsilon_\beta^{(M_\alpha - m_\beta)} \qquad \delta q_{s\ i} = 0$$

$$\delta p_{1\ i} = -\frac{\partial(B_{\alpha\ \beta}^{m_\alpha\ m_\beta}\phi_\alpha^{m_\alpha})}{\partial q_{1\ i}}\varepsilon_\beta^{(M_\alpha - m_\beta)} \qquad \delta p_{s\ i} = -\frac{\partial B_{\alpha\ \beta}^{m_\alpha\ m_\beta}}{\partial q_{s\ i}}\phi_\alpha^{m_\alpha}\varepsilon_\beta^{(M_\alpha - m_\beta)}.$$

Using this equation and the equality

$$\frac{dG}{dt} = \frac{\partial G}{\partial t} + \left[\frac{\partial(B_{\alpha\ \beta}^{m_\alpha\ m_\beta}\phi_\alpha^{m_\alpha})}{\partial q_{1\ i}} q_{2\ i} + \frac{\partial(B_{\alpha\ \beta}^{m_\alpha\ m_\beta}\phi_\alpha^{m_\alpha})}{\partial p_{1\ i}}\dot{p}_{1\ i} + \left(\frac{\partial B_{\alpha\ \beta}^{m_\alpha\ m_\beta}}{\partial q_{s\ i}} q_{s+1\ i} + \frac{\partial B_{\alpha\ \beta}^{m_\alpha\ m_\beta}}{\partial q_{K\ i}}\dot{q}_{K\ i}\right)\phi_\alpha^{m_\alpha}\right]\varepsilon_\beta^{(M_\alpha - m_\beta)}$$

we obtain

$$\delta S = \left[p_{1\ i}\frac{\partial G}{\partial p_{1\ i}} - G\right]_{t_1}^{t_2} + \int_{t_1}^{t_2} dt \left[\frac{\partial G}{\partial t} + \{G, \tilde{H}_T\}\right]. \tag{52}$$

The first term in (52) vanishes due to the boundary conditions on ε_α and their derivatives. The second term of (52), in view of (8), can be written in the form

$$\int_{t_1}^{t_2} dt\, v_\alpha \phi_\alpha^1. \tag{53}$$

and, therefore,

$$\delta S|_{\phi_\alpha^1 \approx 0} = 0.$$

As a result of (49), (53) and $\phi_\alpha^1(q, p(q, \dot{q})) \equiv 0$ we obtain from (52) $\delta S = 0$ in configuration space.

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